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Second-Order Splitting Combined with Orthogonal Cubic Spline Collocation Method for the Kuramoto-Sivashinsky Equation

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Abstract—In this paper, a second-order splitting method is applied to the Kuramoto-Sivashinsky equation and then an orthogonal cubic spline collocation procedure is employed to the approximate resulting system. This semidiscrete method yields a system of differential algebraic equations (DAEs) of index 1. Error estimates in L^2 and L^∞ norms are obtained for the semidiscrete approximation. For the time discretization, the time integrator RADAU5 is used. The results of numerical experiments are presented to validate the theoretical findings.

Keywords—Kuramoto-Sivashinsky equation, Orthogonal spline collocation method, Semidiscrete schemes, Error estimates, Differential algebraic equations, Implicit Runge-Kutta methods.

1. INTRODUCTION

In this paper, we examine the use of the orthogonal spline collocation method in conjunction with a splitting procedure for approximate solution of the Kuramoto-Sivashinsky (K-S) equation

$$u_t + \nu u_{xxxx} + u_{xx} + uu_x = 0, \qquad (x,t) \in I \times (0,T], \tag{1.1}$$

subject to the initial and boundary conditions

$$u(x,0) = u_0(x), \qquad x \in I, \tag{1.2}$$

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$$u(0,t) = u(1,t) = 0, \quad u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad t \in (0,T], \tag{1.3}$$

where ν is a positive constant, $I = (0,1)$ and $0 < T < \infty$.

The K-S equation was derived independently by Kuramoto [1] and Sivashinsky [2] in the late 70's while working, respectively, on turbulence phenomena in chemistry and thermal diffusive instability in laminar flame fronts. It also arises in a variety of applications among which are the modeling of reaction-diffusion systems, flame propagation, and viscous flow problems. Moreover, the K-S equation is also considered to be a prototype of a system with "self-generated" chaos and the large class of generalized Burgers' equations. The well-posedness of the K-S equation has been examined in [3]. For the mathematical theory, physical significance and theoretical studies such as long time behaviour, we refer the reader to [1,2,4,5] and references therein.

The finite difference scheme is applied to the problem (1.1) with periodic boundary conditions and the related error analysis is examined in [6]. In [7], a finite element Galerkin method with extrapolated Crank-Nicolson scheme is discussed and related *a priori* error bounds are derived. A pseudo spectral Fourier method is applied to (1.1) with periodic boundary conditions, and nonlinear stability with convergence is examined in [8]. These papers are aimed at obtaining theoretically sound space and time discretizations.

The rigorous numerical experiments related to the dynamics of the K-S equation are first reported by Nicolaenko *et al.* [9] using pseudo-spectral methods. In [10], Collet *et al.* have studied the analyticity properties of the K-S equation for periodic initial data with period L. The numerical results presented there show strong evidence of analyticity property in a strip around the real axis whose width is independent of L. Now in recent literature [11,12], attention has been focused on the computation of inertial manifolds of dissipative PDEs, in particular, the K-S equation using Galerkin and nonlinear Galerkin methods in space and standard stiff ODE solvers in time. In most cases, a higher order spatial discretization method is coupled with lower order time stepping scheme, except in [11], where different kind of stiff ODE solvers including BDF methods are discussed. We shall in this article use accuracy of order $O(h^4)$ in space and of order $O(k^5)$ in time, where h and k are, respectively, the space and time discretization parameters.

In this paper, we shall discuss the numerical solution of (1.1) using an orthogonal spline collocation(OSC) method. A direct application of this procedure requires at least C^3-piecewise quintic polynomials in spatial direction. For orthogonal collocation method applied to m^{th} (m even) order boundary value problem, we refer the reader to [13]. The use of higher degree C^3-splines increases computational complexity, i.e., it results in a matrix with large size bandwidth and moreover, the computation of the nonlinear term uu_x possesses additional problems. In contrast, we propose here a second-order splitting technique [14] for the K-S equation (1.1) and then apply the orthogonal cubic spline collocation method to the resulting approximate system. To this end, setting

$$v = -u_{xx}, \tag{1.4}$$

the equations (1.1)–(1.3) may be rewritten as

$$u_t - \nu v_{xx} - v + uu_x = 0, \quad v + u_{xx} = 0, \qquad (x,t) \in I \times (0,T], \tag{1.5}$$

subject to the initial and the boundary conditions

$$u(x,0) = u_0(x), \qquad x \in I, \tag{1.6}$$

$$u(0,t) = u(1,t) = 0, \quad v(0,t) = v(1,t) = 0, \qquad t \in (0,T]. \tag{1.7}$$

Clearly, the above system represents a natural splitting of (1.1) into a coupled system for u and v. The OSC method in the space variable using monomial basis functions is then applied to approximate solutions of the system (1.5)–(1.7). Its advantages and popularity (see [15]) over B-splines and the variant Hermite type basis, are due in part to its conceptual simplicity, wide applicability and ease of implementation. It is much superior to B-splines in terms of stability, efficiency, and conditioning of the resulting matrix [16]. Compared to finite element methods (FEM), the calculation of the coefficients of the mass and stiffness matrices determining the approximate solution is very simple, since no integrals need to be evaluated or approximated in this case. Another positive point in favour of our proposed method is that it systematically incorporates both boundary and interface (continuity) conditions. The resulting semidiscrete system is solved by the code RADAU5 [17,18] which is suitable for solving differential algebraic equations (DAEs).

A brief outline of this paper is as follows. In Section 2, we introduce some definitions, notations, preliminaries and describe some properties of the K-S equation. In Section 3, the continuous orthogonal cubic spline collocation method is analyzed and optimal error estimates in L^2 and L^∞ norms for both u and v are derived for the problem (1.5)–(1.7). Finally in Section 4, numerical experiments confirming our theoretical results are described and long time behaviour of the K-S equation is illustrated through numerical results.

2. PRELIMINARIES

In our subsequent analysis, we shall use $L^2(I)$ as a space of square integrable real valued functions with inner product $(.,.)$ and norm $\|.\|$. For a nonnegative integer r, let $H^r(I)$ (in short H^r) be the standard Hilbert Sobolev space with norm $\|.\|_r$. Further, let $H_0^1(I) = \{w \in H^1(I) : w(0) = w(1) = 0\}$. For nonnegative integer m, let $W^{m,p}(I)$ denote a Sobolev space with norm

$$\|u\|_{W^{m,p}(I)} = \begin{cases} \left(\sum_{i=0}^m \|u^{(i)}\|_{L^p(I)}^p\right)^{1/p}, & 1 \le p < \infty, \\ \max_{0\le i\le m} \|v^{(i)}\|_{L^\infty(I)}, & \text{if } p = \infty. \end{cases}$$

Sometimes, we shall use $L^p(H^s)$ as $L^p(0,T;H^s(I))$.

Now the variational formulation of (1.5) is defined as a pair $\{u,v\} : (0,T] \longrightarrow H_0^1 \times H_0^1$ satisfying

$$\begin{aligned} (u_t,z) + \nu(v_x,z_x) - (v,z) + (uu_x,z) &= 0, \\ -(u_x,\omega_x) + (v,\omega) &= 0, \end{aligned} \qquad \forall\, z,\omega \in H_0^1(I). \tag{2.1}$$

Below, we shall derive some properties of the K-S equation. Choose $z = u(.,t)$ and $\omega = \nu v(.,t)$ in the first and second equations of (2.1), respectively. Since $(uu_x,u) = 0$, using the boundary conditions for u, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(.,t)\|^2 = -\nu\|v(.,t)\|^2 + (v,u). \tag{2.2}$$

Apply Young's inequality $|(v,u)| \le \nu\|v(.,t)\|^2 + (1/4\nu)\|u(.,t)\|^2$ to obtain

$$\frac{d}{dt}\|u(.,t)\|^2 \le \frac{1}{2\nu}\|u(.,t)\|^2,$$

from which it follows that

$$\|u(.,t)\| \le \|u_0\|e^{t/4\nu}. \tag{2.3}$$

Note that the above boundedness property is valid only for finite time t. Since $u(0) = u(1) = 0$, by Roll's theorem there exists $s \in (0,1)$ such that $u_x(s) = 0$. Therefore, $|u_x(y)| = |\int_s^y u_{xx}(\tau)\,d\tau|$ and hence, use of Cauchy Schwarz inequality yields

$$\|u_x\|^2 \le \frac{1}{2}\|u_{xx}\|^2 = \frac{1}{2}\|v\|^2.$$

Since $u \in H_0^1(I)$, we use Wirtinger inequality (see [19, pp. 1196–1197]) to obtain

$$\|u\| \le \frac{1}{\pi}\|u_x\|,$$

and therefore,

$$|(u,v)| \le \|u\|\,\|v\| \le \frac{1}{\pi}\|u_x\|\,\|v\| \le \frac{1}{\sqrt{2}\,\pi}\|v\|^2.$$

Substituting in (2.2), it follows that

$$\frac{1}{2}\frac{d}{dt}\|u(.,t)\|^2 \le \left(\frac{1}{\sqrt{2}\,\pi} - \nu\right)\|v(.,t)\|^2,$$

which, in turn, yields

$$\|u(.,t)\| \le \|u(.,s)\|, \qquad 0 \le s \le t \le T, \text{ for } \nu \ge \frac{1}{\sqrt{2}\,\pi}. \tag{2.4}$$

The above result is called monotone property of the K-S equation which is valid for all $t > 0$.

For the orthogonal spline collocation method, let $\Delta = \{x_i\}_{i=1}^{N+1}$ denote a partition of $\overline{I} = [0,1]$ and be such that

$$\Delta:\ 0 = x_1 < x_2 < \cdots < x_{N+1} = 1.$$

Let $I_j = (x_j, x_{j+1})$ with $h_j = x_{j+1} - x_j$, $j = 1,2,\ldots,N$, and $h = \max_{1\le j\le N} h_j$. Assume that this partition is quasi-uniform, i.e., there exists a finite positive number σ such that

$$\max_{1\le j\le N}\left(\frac{h}{h_j}\right) \le \sigma.$$

Define a finite-dimensional subspace $\mathcal{M}_3(\Delta)$ of $H_0^1(I)$ as

$$\mathcal{M}_3(\Delta) = \left\{v \in C^1(\overline{I}):\ v|_{\overline{I}_j} \in P_3,\ j = 1,2,\ldots,N \text{ and } v(0) = v(1) = 0\right\},$$

where P_3 denotes the set of all cubic polynomials. Let $\{\lambda_k\}_{k=1}^2$ denote the roots of the Legendre Polynomial of degree 2 (i.e., $\lambda_1 = 1/2(1-1/\sqrt{3}), \lambda_2 = 1/2(1+1/\sqrt{3})$). These are the nodes of the 2-point Gaussian quadrature rule on the interval I with corresponding weights $w_k = 1/2,\ k = 1,2$. Set

$$\lambda_{jk} = x_j + h_j\,\lambda_k, \qquad j = 1,2,\ldots,N, \quad k = 1,2.$$

For $\varphi, \psi \in C^0(\overline{I})$, we define a discrete innerproduct and its induced norm by

$$\langle \varphi,\psi\rangle = \sum_{j=1}^{N}\langle\varphi,\psi\rangle_j, \qquad |\varphi|_D = \langle\varphi,\varphi\rangle^{1/2},$$

where

$$\langle\varphi,\psi\rangle_j = \frac{h_j}{2}\sum_{k=1}^{2}\varphi(\lambda_{jk})\,\psi(\lambda_{jk}).$$

Since Δ belongs to a quasi-uniform family of partitions, it can be shown (cf. [20]) that there exist positive constants C_1 and C_2 such that for any $\psi \in \mathcal{M}_3(\Delta)$

$$C_1\ |\psi|_D^2 \le \|\psi\|_{L^2}^2 \le C_2|\psi|_D^2. \tag{2.5}$$

For a smooth function f on I, the error $Q(f)$ in 2-point composite Gaussian quadrature is given by

$$Q(f) = \int_0^1 f\,dx - \langle f,1\rangle,$$

and using the Peano Kernel Theorem, we obtain

$$|Q(f)| \le C\sum_{j=1}^{N} h_j^4\left\|f^{(4)}\right\|_{L^1(I_j)}.$$

The following lemma is proved in [20].

LEMMA 2.1. *For* $\alpha,\ \beta \in \mathcal{M}_3(\Delta)$,

$$-\langle \alpha, \beta'' \rangle = (\alpha', \beta') + \frac{1}{1080} \sum_{j=1}^{N} \alpha_j^{(3)}\ \beta_j^{(3)}\ h_j^5 = -\langle \beta, \alpha'' \rangle,$$

where $\alpha_j^{(3)}$ *(respectively,* $\beta_j^{(3)}$*), the third derivative is constant on* $\overline{I}_j$.

Note that when $\beta = \alpha$ with $\alpha \in \mathcal{M}_3(\Delta)$, we have

$$\|\alpha'\|_{L^2}^2 \le -\langle \alpha, \alpha'' \rangle.$$

Here we state the Jensen's inequality, which will be used throughout our analysis (see [21]).

LEMMA 2.2. *For a bounded interval* $E \subset R$ *with length* $|E|$, *let* $v \in W^{m,p}(E)$ *for* $p \in [1,\infty]$. *Then there exists a constant* C, *depending only on* m *such that for any* d *satisfying* $0 < d \le |E| \le 1$

$$\|v\|_{W^{s,p}(E)} \le C \left[d^{m-s} \|v\|_{W^{m,p}(E)} + d^{-s} \|v\|_{L^p(E)} \right], \qquad 0 \le s \le m-1.$$

3. THE CONTINUOUS-TIME ORTHOGONAL CUBIC SPLINE COLLOCATION METHOD

The continuous-time orthogonal cubic spline collocation approximation to the solution $\{u, v\}$ of (1.1) is a pair of differentiable maps $\{U,\ V\} :\ [0,T] \longrightarrow \mathcal{M}_3(\Delta) \times \mathcal{M}_3(\Delta)$ such that for $j = 1, 2, \dots, N$ and $k = 1, 2$

$$U_t(\lambda_{jk}, t) - \nu V_{xx}(\lambda_{jk}, t) - V(\lambda_{jk}, t) + U(\lambda_{jk}, t) U_x(\lambda_{jk}, t) = 0, \qquad t \in (0,T], \quad (3.1)$$

$$-U_{xx}(\lambda_{jk}, t) = V(\lambda_{jk}, t), \qquad t \in (0,T], \quad (3.2)$$

with a suitably chosen initial approximation $U(x, 0)$, which will be defined later.

Following [20], the corresponding discrete Galerkin formulation can be written as

$$\langle U_t, z \rangle - \nu \langle V_{xx}, z \rangle - \langle V, z \rangle + \langle U\ U_x, z \rangle = 0, \qquad \forall\, z \in \mathcal{M}_3(\Delta), \quad (3.3)$$

$$-\langle U_{xx}, w \rangle = \langle V, w \rangle, \qquad \forall\, w \in \mathcal{M}_3(\Delta). \quad (3.4)$$

Note that the consistent initial condition $V(0)$ can be determined from (3.4) at $t = 0$, i.e., $V(0)$ satisfies

$$\langle V(x,0), w \rangle = -\langle U_{xx}(x,0), w \rangle, \qquad w \in \mathcal{M}_3(\Delta). \quad (3.5)$$

Since $\mathcal{M}_3(\Delta)$ is a finite-dimensional space, problems (3.1),(3.2) leads to a system of nonlinear differential algebraic equations of index one. For the solvability of the present system, it is easy to verify the sufficient conditions A(1)–A(4) in [22] and hence, existence of a unique local solution is guaranteed. The global existence for all $t \in [0,T]$ in a neighborhood of the exact solution will follow from *a priori* estimates in the last part of the proof of Theorem 3.1.

In the error analysis, we shall use the intermediate projections as differentiable maps $\{\hat{U}, \hat{V}\}$: $[0,T] \longrightarrow \mathcal{M}_3(\Delta) \times \mathcal{M}_3(\Delta)$ satisfying

$$-\nu \left\langle \left(v - \hat{V}\right)_{xx}, z \right\rangle + \left\langle \left(v - \hat{V}\right), z \right\rangle = 0, \qquad \forall\, z \in \mathcal{M}_3(\Delta), \quad (3.6)$$

$$\left\langle \left(u - \hat{U}\right)_{xx}, \omega \right\rangle = 0, \qquad \forall\, \omega \in \mathcal{M}_3(\Delta). \quad (3.7)$$

Let $\eta = u - \hat{U}$ and $\rho = v - \hat{V}$. The following estimates of η and ρ are proved in Theorem 2.2 of [15].

LEMMA 3.1. *Let $u, v \in C^1(\bar{I})$ be such that $u, v \in H^6(I_j)$, $j = 1, 2, \ldots, N$. Further, let $\hat{U}$ and $\hat{V}$ be solutions of (3.6) and (3.7), respectively. Then for $t \in (0, T]$*

$$|\eta|_D + |\eta_t|_D \le \|\eta\|_{L^\infty(I)} + \|\eta_t\|_{L^\infty(I)} \le Ch^4 \left(\|u\|_{H^6(I)} + \|u_t\|_{H^6(I)}\right), \tag{3.8}$$

and

$$|\rho|_D + |\rho_t|_D \le \|\rho\|_{L^\infty(I)} + \|\rho_t\|_{L^\infty(I)} \le Ch^4 \left(\|v\|_{H^6(I)} + \|v_t\|_{H^6(I)}\right). \tag{3.9}$$

For our error analysis, we first split the error $e = u - U$ and $E = v - V$, respectively, as $e = (u - \hat{U}) - (U - \hat{U}) = \eta - \theta$ and $E = (v - \hat{V}) - (V - \hat{V}) = \rho - \zeta$.

THEOREM 3.1. *Let $u \in L^\infty(H^8)$, $u_t \in L^2(H^8)$ with $u_0 \in H^8(I)$. Further, let U and V be the solutions of (3.1),(3.2). Let $U(0) = \hat{U}(0)$ so that $\theta(0) = 0$. Then for sufficiently small h*

$$\|u - U\|_{L^\infty(L^2)} + \|v - V\|_{L^\infty(L^2)} \le Ch^4 \left\{\|u_0\|_{H^8} + \|u\|_{L^\infty(H^8)} + \|u_t\|_{L^2(H^8)}\right\}.$$

PROOF. Since the estimates of η and ρ are known from the Lemma 3.1, it is sufficient to estimate θ and ζ. Form a discrete innerproduct between the first and second equations of (1.5) with z and $\omega \in \mathcal{M}_3(\Delta)$, respectively. Then subtract the resulting equations from (3.3),(3.4) and use (3.6),(3.7) to obtain

$$\langle \theta_t, z\rangle - \nu\langle \zeta_{xx}, z\rangle - \langle \zeta, z\rangle = \langle \eta_t, z\rangle - 2\langle \rho, z\rangle + \langle uu_x - UU_x, z\rangle, \qquad \forall\, z \in \mathcal{M}_3(\Delta) \tag{3.10}$$

$$\langle \theta_{xx}, w\rangle = \langle \rho - \zeta, w\rangle, \qquad \forall\, w \in \mathcal{M}_3(\Delta). \tag{3.11}$$

Now assume temporarily that for sufficiently small h

$$\|\theta(t)\|_{L^\infty(I)} \le h, \qquad t \in [0, T]. \tag{3.12}$$

Then using the smoothness of u and Lemma 3.1, it follows that

$$\|U\|_{L^\infty(L^\infty)} \le C.$$

Now choosing $z = \theta$ and $w = \nu\zeta$ in (3.10) and (3.11), respectively, we add the resulting equations to obtain

$$\frac{1}{2}\frac{d}{dt}|\theta|_D^2 \le C\left\{|\rho|_D^2 + |\eta_t|_D^2 + |\zeta|_D^2 + |\theta|_D^2\right\} + |\langle uu_x - UU_x, \theta\rangle|. \tag{3.13}$$

To estimate the nonlinear term in this inequality, we rewrite it as

$$\langle uu_x - UU_x, \theta\rangle = \langle u\eta_x, \theta\rangle + \left\langle -\eta\eta_x + \eta u_x - \theta\hat{U}_x - U\theta_x, \theta\right\rangle.$$

For the first term, we note that

$$\langle u\eta_x, \theta\rangle = \int_0^1 u\eta_x\theta\, dx - Q(u\eta_x\theta),$$

and hence, using integration by parts we have

$$\langle u\eta_x, \theta\rangle = -\int_0^1 \eta(u_x\theta + u\theta_x)\, dx - Q(u\eta_x\theta). \tag{3.14}$$

For the quadrature error

$$|Q(u\eta_x\theta)| \le C\sum_{j=1}^{N} h_j^4 \int_{I_j} \left|(u\eta_x\theta)^{(4)}\right| dx. \tag{3.15}$$

In view of the Jensen's inequality (Lemma 2.2), the following estimates hold true

$$\|\eta\|_{H^s(I_j)} \leq C\left[h_j^{-s}\|\eta\|_{L^2(I_j)} + h_j^{4-s}\|\eta\|_{H^4(I_j)}\right], \qquad s = 1,2,3.$$

Since $\hat{U} \in \mathcal{M}_3(\Delta)$, it follows from the definition of $\|.\|_{H^m(I_j)}-$ norm that

$$\|\eta\|_{H^m(I_j)} \leq \left[\|\eta\|_{H^3(I_j)} + \|u^{(m)}\|_{L^2(I_j)}\right], \qquad m = 4,5.$$

Note that for $s = 3$ and for small h

$$\|\eta\|_{H^3(I_j)} \leq C\left(h_j^{-3}\|\eta\|_{L^2(I_j)} + h_j\left\|u^{(4)}\right\|_{L^2(I_j)}\right).$$

For $s = 1,2$, the result follows similarly, and thus for $s = 1,2,3$

$$\|\eta\|_{H^s(I_j)} \leq C\left(h_j^{-s}\|\eta\|_{L^2(I_j)} + h_j^{4-s}\left\|u^{(4)}\right\|_{L^2(I_j)}\right).$$

Again use of the Jensen's inequality with $\chi \in \mathcal{M}_3(\Delta)$(i.e., $\chi^{(4)} = 0$) yields

$$\|\chi\|_{H^s(I_j)} \leq Ch_j^{-s}\|\chi\|_{L^2(I_j)}, \qquad s = 1,2,3.$$

To obtain a bound for (3.15), it is sufficient to estimate the following three typical terms. Apply the above estimates to obtain

$$\begin{aligned}\sum_{j=1}^{N} h_j^4 \int_{I_j} |u|\left|\eta^{(5)}\right||\theta|\,dx &\leq C\sum_{j=1}^{N} h_j^4\left\|u^{(5)}\right\|_{L^2(I_j)}\|\theta\|_{L^2(I_j)} \\ &\leq C\left[h^8\|u\|^2_{H^5(I)} + \|\theta\|^2_{L^2(I)}\right],\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{N} h_j^4 \int_{I_j} |u|\left|\eta^{(3)}\right|\left|\theta^{(2)}\right|\,dx &\leq C\sum_{j=1}^{N} h_j^4\|\eta\|_{H^3(I_j)}\|\theta_x\|_{H^1(I_j)} \\ &\leq C\sum_{j=1}^{N}\left[\|\eta\|_{L^2(I_j)} + h_j^4\left\|u^{(4)}\right\|_{L^2(I_j)}\right]\|\theta_x\|_{L^2(I_j)} \\ &\leq C\left[\|\eta\|^2_{L^2(I)} + h^8\|u\|^2_{H^4(I)}\right] + \|\theta_x\|^2_{L^2(I)},\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{N} h_j^4 \int_{I_j} |u|\left|\eta^{(2)}\right|\left|\theta^{(3)}\right|\,dx &\leq C\sum_{j=1}^{N} h_j^4\|\eta\|_{H^2(I_j)}\|\theta_x\|_{H^2(I_j)} \\ &\leq C\sum_{j=1}^{N}\left[\|\eta\|_{L^2(I_j)} + h_j^4\left\|u^{(4)}\right\|_{L^2(I_j)}\right]\|\theta_x\|_{L^2(I_j)} \\ &\leq C\left[\|\eta\|^2_{L^2(I)} + h^8\|u\|^2_{H^4(I)}\right] + \|\theta_x\|^2_{L^2(I)}.\end{aligned}$$

Altogether, (3.15) becomes

$$|Q(u\eta_x\theta)| \leq C(u)\left[\|\eta\|^2_{L^2(I)} + \|\theta\|^2_{L^2(I)} + \|\theta_x\|^2_{L^2(I)} + h^8\|u\|^2_{H^5(I)}\right].$$

To estimate $\|\theta_x\|$, choose $w = -\theta$ in (3.11) to obtain

$$\|\theta_x\|^2 \le -\langle \theta_{xx}, \theta \rangle \le C\left(|\zeta|_D + |\rho|_D\right)\|\theta\|_{L^2}$$

and hence, use of Poincaré inequality i.e., $\|\theta\| \le (1/\sqrt{2})\|\theta_x\|$ as $\theta \in H_0^1(I)$ yields

$$\|\theta_x\| \le C\left(|\zeta|_D + |\rho|_D\right). \tag{3.16}$$

Thus, the nonlinear term in (3.13) is bounded by

$$|\langle uu_x - UU_x, \theta\rangle| \le C\left[\|\eta\|^2_{L^\infty(I)} + \|\rho\|^2_{L^\infty(I)} + |\zeta|^2_D + h^8\|u\|^2_{H^5(I)}\right].$$

Substituting the above estimates in (3.13), use Lemma 3.1 and the estimate $\|\theta_x\|$ to obtain

$$\frac{1}{2}\frac{d}{dt}|\theta|^2_D \le C\left\{|\zeta|^2_D + h^8\left[\|u\|^2_{H^6(I)} + \|u_t\|^2_{H^6(I)} + \|v\|^2_{H^6(I)}\right]\right\}. \tag{3.17}$$

To estimate $|\zeta|^2_D$, we first differentiate the equation (3.11) with respect to t to have

$$\langle \theta_{xxt}, w\rangle = \langle \rho_t - \zeta_t, w\rangle, \qquad \forall w \in \mathcal{M}_3(\Delta).$$

Setting $w = \nu\zeta$ in the above equation and $z = \theta_t$ in (3.10), add the resulting equations and use Lemma 2.1 to obtain

$$|\theta_t|^2_D + \frac{\nu}{2}\frac{d}{dt}|\zeta|^2_D \le C\{|\rho|^2_D + |\rho_t|^2_D + |\eta_t|^2_D + |\rho|^2_D + |\theta|^2_D + |\zeta|^2_D + |\theta_t|^2_D\} + |\langle uu_x - UU_x, \theta_t\rangle|. \tag{3.18}$$

The nonlinear term may be rewritten as

$$\langle uu_x - UU_x, \theta_t\rangle = \langle u\eta_x, \theta_t\rangle + \left\langle -\eta\eta_x + \eta u_x - \theta\hat{U}_x - U\theta_x, \theta_t\right\rangle. \tag{3.19}$$

For the first term, we note that

$$\begin{aligned}\langle u\eta_x, \theta_t\rangle &= \frac{d}{dt}\langle u\eta_x, \theta\rangle - \langle (u\eta_x)_t, \theta\rangle \\ &= -\frac{d}{dt}Q(u\eta_x\theta) + Q((u\eta_x)_t\theta) - \frac{d}{dt}\int_0^1 \eta(u\theta)_x\,dx + \int_0^1 (\eta_t(u\theta)_x + \eta(u_t\theta)_x)\,dx.\end{aligned}$$

Integrating in time from 0 to t and using $\theta(0) = 0$ and the estimate of $\|\theta_x\|$, we now proceed as in the case of estimate of (3.14) to obtain

$$\begin{aligned}\int_0^t \langle u\eta_x, \theta_t\rangle\,d\tau &\le C\left\{|\zeta|^2_D + h^8\left[\|u\|^2_{H^6} + \|v\|^2_{H^6}\right]\right\} \\ &\quad + C\int_0^t \left\{|\zeta|^2_D + h^8\left[\|u\|^2_{H^6} + \|u_t\|^2_{H^6} + \|v\|^2_{H^6}\right]\right\}d\tau,\end{aligned}$$

and hence,

$$\begin{aligned}\int_0^t |\langle uu_x - UU_x, \theta_t\rangle|\,d\tau &\le C\left\{|\zeta|^2_D + h^8\left[\|u\|^2_{H^6} + \|v\|^2_{H^6}\right]\right\} \\ &\quad + C\int_0^t \left\{|\zeta|^2_D + |\theta_t|^2_D + h^8\left[\|u\|^2_{H^6} + \|u_t\|^2_{H^6} + \|v\|^2_{H^6}\right]\right\}d\tau.\end{aligned}$$

Now integrate (3.17) and (3.18) from 0 to t, with $t \le T$ and add the resulting inequalities. An application of Gronwall's Lemma yields

$$\int_0^t |\theta_t|^2_D d\tau + |\theta|^2_D + |\zeta|^2_D \le C\left\{|\zeta(0)|^2_D + h^8\left[\|v_t\|^2_{L^2(H^6)} + \|u\|^2_{L^\infty(H^6)} + \|u_t\|^2_{L^2(H^6)} + \|v\|^2_{L^\infty(H^6)}\right]\right\}.$$

In order to estimate $\zeta(0)$, the equation (3.11) at $t = 0$ becomes (as $\theta(0) = 0$)

$$|\zeta(0)|_D \leq |\rho(0)|_D \leq Ch^4 \|v_0\|_{H^6} \leq Ch^4 \|u_0\|_{H^8}.$$

Therefore,

$$\|\theta_t\|_{L^2(L^2)} + \|\theta\|_{L^\infty(L^2)} + \|\zeta\|_{L^\infty(L^2)} \leq Ch^4 \left\{ \|u_0\|_{H^8} + \|u\|_{L^\infty(H^8)} + \|u_t\|_{L^2(H^8)} \right\}.$$

Apply the triangle inequality to complete the proof. For the temporary assumption (3.12), a use of the inverse hypotheses for θ yields

$$\|\theta(t)\|_{L^\infty(I)} \leq Ch^{-1/2} \|\theta(t)\| \leq Ch^{7/2} \leq h,$$

for sufficiently small h. Since the system (3.3),(3.4) is a finite system of differential and algebraic equation of index one, we can assume that (3.12) is satisfied for small $t > 0$. Suppose it fails to hold for some $t \in [0, T]$ and let

$$\tau = \inf\{t \in (0, T] \mid (3.12) \text{ fails to hold}\}.$$

Then by continuity, we have

$$h = \|\theta(\tau)\|_{L^\infty(I)} \leq Ch^{-1/2} \|\theta(\tau)\| \leq Ch^{7/2}.$$

This leads to a contradiction, since h is sufficiently small. Thus, the assumption (3.12) is valid for $t \in [0, T]$. ∎

In the following theorem, we shall prove L^q, $1 \leq q \leq \infty$ estimates for $u - U$.

THEOREM 3.2. *Let all the assumptions of the Theorem 3.1 hold. Then for sufficiently small h*

$$\|u - U\|_{L^\infty(L^q)} \leq Ch^4 \left\{ \|u_0\|_{H^8} + \|u\|_{L^\infty(H^8)} + \|u_t\|_{L^2(H^8)} \right\}, \qquad 1 \leq q \leq \infty.$$

PROOF. Using Theorem 3.1 and Lemma 3.1 in (3.16), we obtain

$$\|\theta_x\|_{L^2(I)} \leq Ch^4 \left\{ \|u_0\|_{H^8} + \|u\|_{L^\infty(H^8)} + \|u_t\|_{L^2(H^8)} \right\}. \tag{3.20}$$

The superconvergence result for θ_x along with Sobolev imbedding theorem for one-dimensional case yields an estimate for $\|\theta\|_{L^q}$. The triangle inequality with estimates of $\|\eta\|_\infty$ completes the rest of the proof. ∎

REMARK 3.1.

(i) It is not difficult to extend the above analysis to the problem (1.1) with periodic boundary conditions. In this case, the inequality (2.4) is valid for all $\nu \geq 1/4\pi^2$, (see [6]). With a slight modification of the intermediate projection (3.7), i.e., let $\hat{U}$ be a solution of

$$\left\langle -\left(u - \hat{U}\right)_{xx} + \lambda\left(u - \hat{U}\right), \omega\right\rangle = 0, \qquad \omega \in \mathcal{M}_{\text{per}}(\Delta),$$

where $\mathcal{M}_{per}(\Delta) = \{\phi \in C^1(\bar{I}) : \phi|_{\bar{I}_j} \in P_3,\ j = 1, \ldots, N, \phi \text{ periodic of period } 1\}$. The error estimates in Theorems 3.1 and 3.2 can be easily derived. Similarly these results can be carried out for the problem (1.1) with the following boundary conditions

$$u_x(0,t) = u_x(1,t) = 0, \qquad u_{xxx}(0,t) = u_{xxx}(1,t) = 0, \qquad t \in (0, T].$$

(ii) The present analysis can be easily extended to the case of C^1-piecewise polynomials of degree $r \geq 3$ with appropriate changes in the quadrature error and the Lemma 2.1. For example, instead of Lemma 2.1, if we apply Lemma 2.1 of [15] (see p. 359), then after the suitable modification of the nonlinear term, it is easy to derive the following result

$$\|u - U\|_{L^\infty(L^q)} + \|v - V\|_{L^\infty(L^2)} \leq Ch^{r+1} \left\{ \|u_0\|_{H^{r+5}} + \|u\|_{L^\infty(H^{r+5})} + \|u_t\|_{L^2(H^{r+5})} \right\},$$
$$1 \leq q \leq \infty.$$

4. NUMERICAL EXPERIMENTS

In this section, we shall present some numerical results, where the problem (1.5)–(1.7) is solved using orthogonal cubic spline collocation method. The approximate solution is defined as a pair of differentiable maps $\{U,V\} : [0,T] \to \mathcal{M}_3(\Delta) \times \mathcal{M}_3(\Delta)$ satisfying (3.1),(3.2). Using the monomial basis functions, we represent U and V as

$$U(x,t) = \sum_{l=1}^{4} U_{j,l}(t)\frac{(x-x_j)^{l-1}}{(l-1)!}, \qquad x \in \bar{I}_j \tag{4.1}$$

and

$$V(x,t) = \sum_{l=1}^{4} V_{j,l}(t)\frac{(x-x_j)^{l-1}}{(l-1)!}, \qquad x \in \bar{I}_j, \tag{4.2}$$

where,

$$\begin{aligned} U_{j,1}(t) &= U(x_j,t), & U_{j,2}(t) &= U_x(x_j,t), \\ U_{j,3}(t) &= U_{xx}\left(x_j^+,t\right), & U_{j,4}(t) &= U_{xxx}\left(x_j^+,t\right), \end{aligned} \qquad j = 1,2,\dots,N, \tag{4.3}$$

and similarly for $V_{j,l}$. In order to accommodate the boundary conditions, we define

$$\begin{aligned} U_{N+1,1}(t) &= U(x_{N+1},t), & V_{N+1,1}(t) &= V(x_{N+1},t), \\ U_{N+1,2}(t) &= U_x(x_{N+1},t), & V_{N+1,2}(t) &= V_x(x_{N+1},t). \end{aligned} \tag{4.4}$$

Using (4.1),(4.2) in (3.1),(3.2), we obtain a system of differential algebraic equations (DAEs)

$$\begin{aligned} \sum_{l=1}^{4} \dot{U}_{j,l}\frac{(h_j\lambda_k)^{l-1}}{(l-1)!} &= \nu(V_{j,3} + h_j\lambda_k V_{j,4}) + \sum_{l=1}^{4} V_{j,l}\frac{(h_j\lambda_k)^{l-1}}{(l-1)!} \\ &\quad - \left(\sum_{l=1}^{4} U_{j,l}\frac{(h_j\lambda_k)^{l-1}}{(l-1)!}\right)\sum_{l=2}^{4} U_{j,l}\frac{(h_j\lambda_k)^{l-2}}{(l-2)!}, \end{aligned} \tag{4.5}$$

$$\sum_{l=1}^{4} V_{j,l}\frac{(h_j\lambda_k)^{l-1}}{(l-1)!} + (U_{j,3} + h_j\lambda_k U_{j,4}) = 0, \tag{4.6}$$

for $j = 1,2,\dots,N,\ \ k = 1,2$, where $\dot{U}_{j,k}(t) = \frac{d}{dt}U_{j,k}(t)$. The C^0 and C^1 continuity conditions on U and V require that for j=1,2, ... ,N

$$\begin{aligned} U_{j+1,1} &= U_{j,1} + h_j\, U_{j,2} + \frac{h_j^2}{2!}\, U_{j,3} + \frac{h_j^3}{3!} U_{j,4}, \\ V_{j+1,1} &= V_{j,1} + h_j\, V_{j,2} + \frac{h_j^2}{2!}\, V_{j,3} + \frac{h_j^3}{3!} V_{j,4}, \\ U_{j+1,2} &= U_{j,2} + h_j\, U_{j,3} + \frac{h_j^2}{2!}\, U_{j,4}, \\ V_{j+1,2} &= V_{j,2} + h_j\, V_{j,3} + \frac{h_j^2}{2!}\, V_{j,4}. \end{aligned} \tag{4.7}$$

The boundary conditions (1.7), in view of (4.4) yield the following set of equations

$$\begin{aligned} U_{1,1} &= 0, & U_{N+1,1} &= 0, \\ V_{1,1} &= 0, & V_{N+1,1} &= 0. \end{aligned} \tag{4.8}$$

Rewriting the equations (4.5)–(4.8) in matrix form, we obtain

$$\mathbf{M}\,\mathbf{W}' = \mathbf{F}(t,\mathbf{W}), \tag{4.9}$$

which is of order $8N+4$ with

$$\mathbf{W} = \left[\mathbf{W}_1^{1\top}, \mathbf{W}_1^{2\top}, \mathbf{W}_2^{1\top}, \mathbf{W}_2^{2\top}, \ldots, \mathbf{W}_N^{1\top}, \mathbf{W}_N^{2\top}, \mathbf{W}_{N+1}^{1\top}\right]^\top,$$

where

$$\mathbf{W}_j^1 = [U_{j,1}, V_{j,1}, U_{j,2}, V_{j,2}]^\top, \qquad \mathbf{W}_j^2 = [U_{j,3}, V_{j,3}, U_{j,4}, V_{j,4}]^\top, \qquad j = 1, \ldots, N,$$

$$\mathbf{W}_{N+1}^1 = [U_{N+1,1}, V_{N+1,1}, U_{N+1,2}, V_{N+1,2}]^\top$$

and the singular constant mass matrix $\mathbf{M}$ is of the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{0}_{24} & & & & & & & \\ \mathbf{A}_1 & \mathbf{B}_1 & & & & & & \\ & \mathbf{0}_{44} & & & & & & \\ & & \mathbf{A}_2 & \mathbf{B}_2 & & & & \\ & & & \mathbf{0}_{44} & & & & \\ & & & & \ddots & & & \\ & & & & & \mathbf{A}_N & \mathbf{B}_N & \mathbf{0}_{44} \\ & & & & & & \mathbf{0}_{44} & \mathbf{0}_{44} \\ & & & & & & & \mathbf{0}_{24} \end{bmatrix}. \tag{4.10}$$

Here $\mathbf{A}_j$ and $\mathbf{B}_j$ are 4×4 matrices corresponding to the coefficients of the derivative terms in (4.5),(4.6) and are of the form

$$\mathbf{A}_j = \begin{bmatrix} 1 & 0 & h_j\lambda_1 & 0 \\ 1 & 0 & h_j\lambda_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B}_j = \begin{bmatrix} \frac{(h_j\lambda_1)^2}{2!} & 0 & \frac{(h_j\lambda_1)^3}{3!} & 0 \\ \frac{(h_j\lambda_2)^2}{2!} & 0 & \frac{(h_j\lambda_2)^3}{3!} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and 0_{24} and 0_{44} are 2×4 and 4×4 zero matrices, respectively.

The initial value $\mathbf{W}(0)$ is found out from the intermediate projection (3.7) at $t = 0$ and the consistency condition (3.5). Since $U(0) = \hat{U}(0)$ from the assumption of the Theorem 3.1, it follows from (3.7)

$$U_{xx}(\lambda_{jk}, 0) = u_{0xx}(\lambda_{jk}), \qquad j = 1, \ldots, N, \quad k = 1, 2.$$

Now along with the continuity conditions at the interior nodal points x_j and the boundary conditions, we obtain a linear system of $4N+2$ equations

$$\mathbf{R}\,\mathbf{U} = \mathbf{U}_0, \tag{4.11}$$

where $\mathbf{R}$ is $(4N+2) \times (4N+2)$ block diagonal nonsingular matrix of the form

$$\mathbf{R} = \begin{bmatrix} & \mathbf{I}_1 & & & & & & \\ \mathbf{X}_1 & \mathbf{Y}_1 & \mathbf{0}_{22} & & & & & \\ \mathbf{C}_1 & \mathbf{D}_1 & \mathbf{I}_{22} & & & & & \\ & & \mathbf{X}_2 & \mathbf{Y}_2 & \mathbf{0}_{22} & & & \\ & & \mathbf{C}_2 & \mathbf{D}_2 & \mathbf{I}_{22} & & & \\ & & & & & \ddots & & \\ & & & & & \mathbf{X}_N & \mathbf{X}_N & \mathbf{0}_{22} \\ & & & & & \mathbf{C}_N & \mathbf{D}_N & \mathbf{I}_{22} \\ & & & & & & & \mathbf{I}_1 \end{bmatrix}.$$

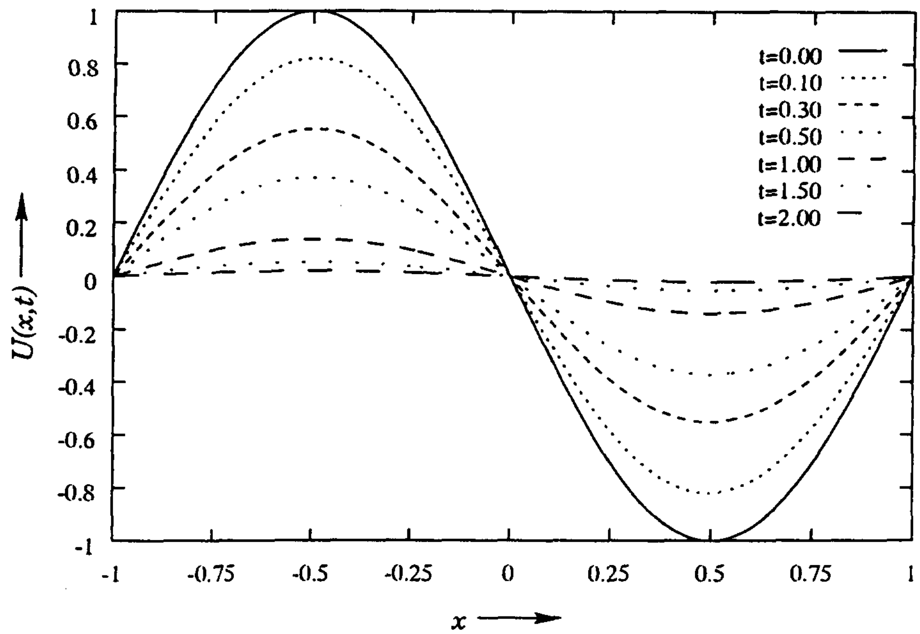


Figure 1. The profile of $U(x,t)$ vs x for $\nu = 1.2/\pi^2$.

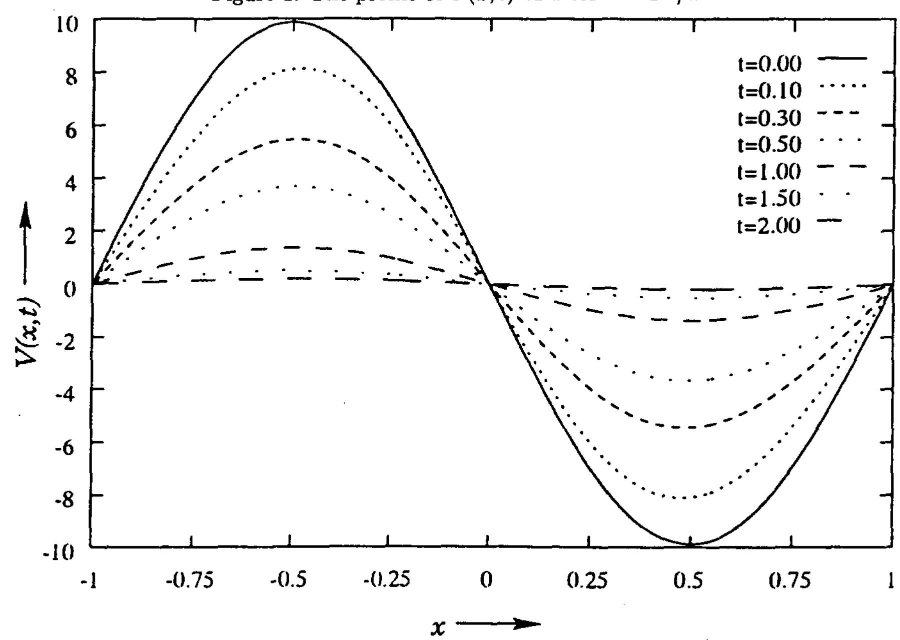


Figure 2. The profile of $V(x,t)$ vs x for $\nu = 1.2/\pi^2$.

Here

$$\mathbf{X}_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{Y}_j = \begin{bmatrix} 1 & h_j\lambda_1 \\ 1 & h_j\lambda_2 \end{bmatrix}, \quad \mathbf{C}_j = \begin{bmatrix} -1 & -h_j \\ 0 & -1 \end{bmatrix}, \quad \mathbf{D}_j = \begin{bmatrix} -\dfrac{h_j^2}{2!} & -\dfrac{h_j^3}{3!} \\ -h_j & -\dfrac{h_j^2}{2!} \end{bmatrix},$$

I_{22} is the identity matrix of order 2×2, $\mathbf{0}_{22}$ is the zero matrix of order 2×2 and $\mathbf{I}_1 = (1, 0)$ Further, the vectors $\mathbf{U}_0$ and $\mathbf{U}$ are of the form

$$\mathbf{U}_0 = [0, u_{0xx}(\lambda_{11}), u_{0xx}(\lambda_{12}), 0, 0, u_{0xx}(\lambda_{21}), u_{0xx}(\lambda_{22}), \dots, u_{0xx}(\lambda_{N1}), u_{0xx}(\lambda_{N2}), 0, 0, 0]^\top$$

and

$$\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_N, \mathbf{U}_{N+1}]^\top \in R^{4N+2}$$

with $\mathbf{U}_j = [U_{j,1}, U_{j,2}, U_{j,3}, U_{j,4}]$, $j = 1, \dots, N$ and $\mathbf{U}_{N+1} = [U_{N+1,1}, U_{N+1,2}]$.

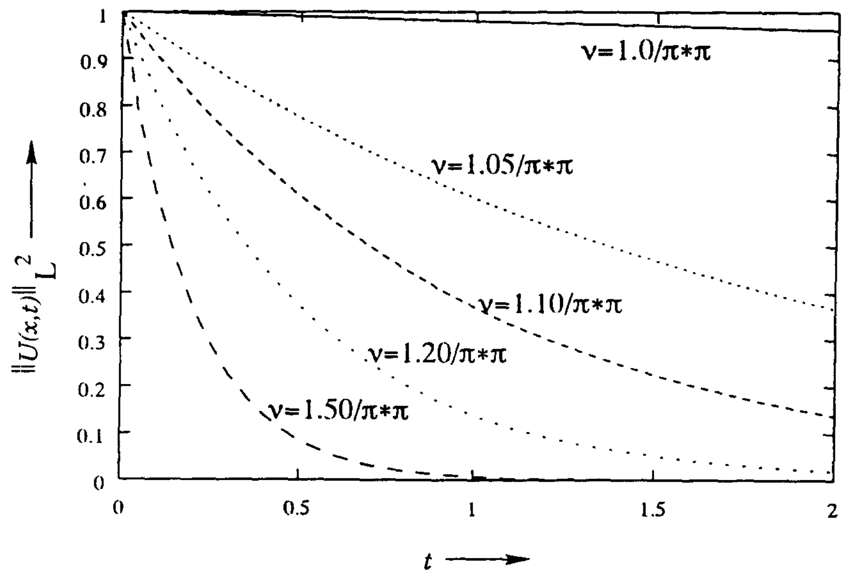


Figure 3. The L^2 norm of the solution for $\nu \geq 1.0/\pi^2$.

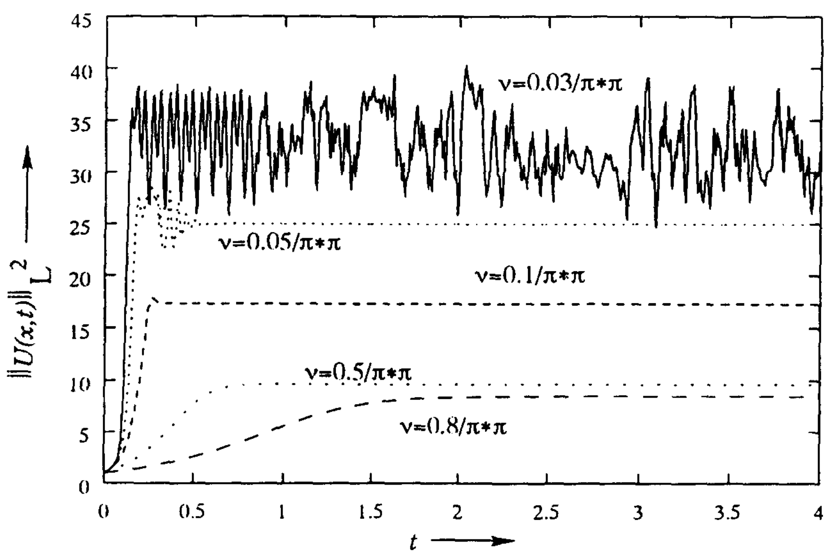


Figure 4. The L^2 norm of the solution for $\nu < 1.0/\pi^2$.

Similarly for $V(x,0)$, we obtain from (3.5)

$$V(\lambda_{jk},0) = -U_{xx}(\lambda_{jk},0), \qquad j=1,\dots,N, \quad k=1,2.$$

which along with continuity and boundary conditions yield a system of $4N+2$ equations

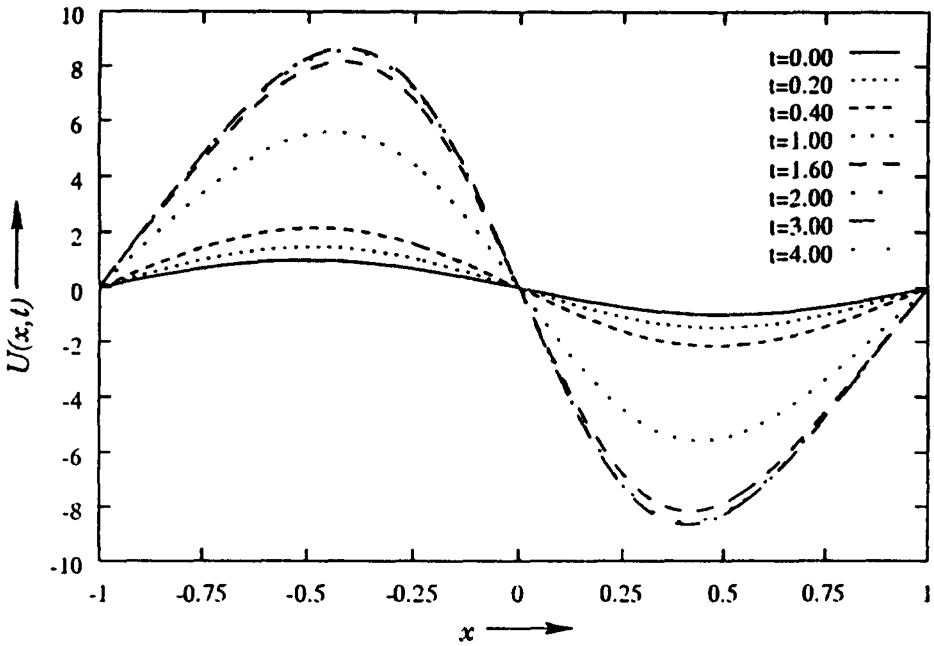
$$\tilde{\mathbf{R}}\,\mathbf{V} = \mathbf{V}_0, \tag{4.12}$$

where

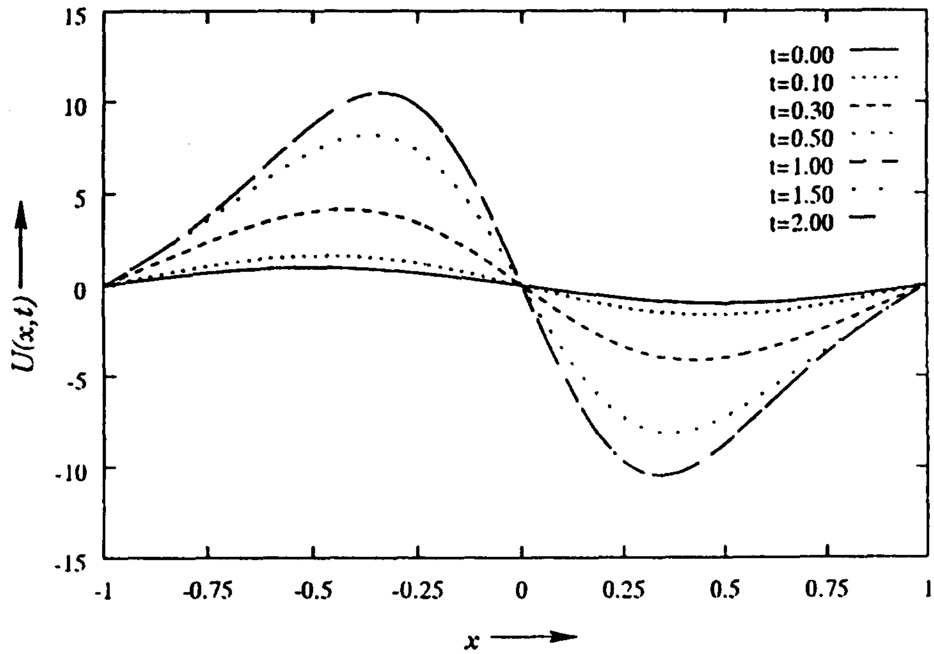
$$\mathbf{V}_0 = [0, -U_{1,3} - h_1\lambda_1 U_{1,4}, -U_{1,3} - h_1\lambda_2 U_{1,4}, 0, 0, -U_{2,3} - h_2\lambda_1 U_{2,4}, -U_{2,3} \\ - h_2\lambda_2 U_{2,4}, 0, 0, \dots, -U_{N,3} - h_N\lambda_1 U_{N,4}, -U_{N,3} - h_N\lambda_2 U_{N,4}, 0, 0, 0]^\top,$$

and all the submatrices in $\tilde{\mathbf{R}}$ are similar to $\mathbf{R}$ in (4.11) except for

$$\mathbf{X}_j = \begin{bmatrix} 1 & h_j\lambda_1 \\ 1 & h_j\lambda_2 \end{bmatrix}, \qquad \mathbf{Y}_j = \begin{bmatrix} \dfrac{(h_j\lambda_1)^2}{2!} & \dfrac{(h_j\lambda_1)^3}{3!} \\ \dfrac{(h_j\lambda_2)^2}{2!} & \dfrac{(h_j\lambda_2)^3}{3!} \end{bmatrix}.$$



(a) When $\nu = 0.8/\pi * \pi$.

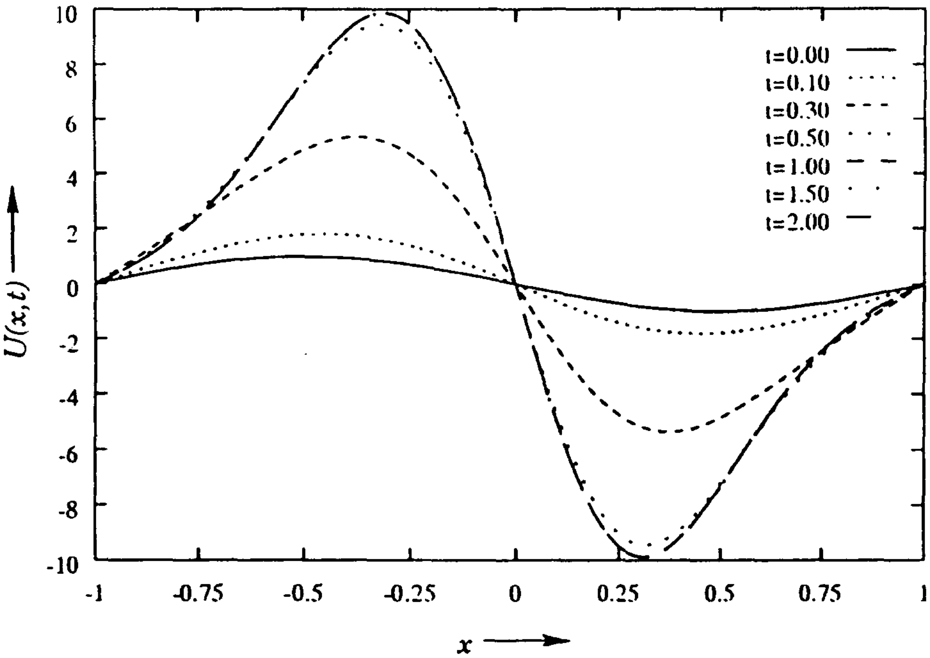


(b) When $\nu = 0.5/\pi * \pi$.

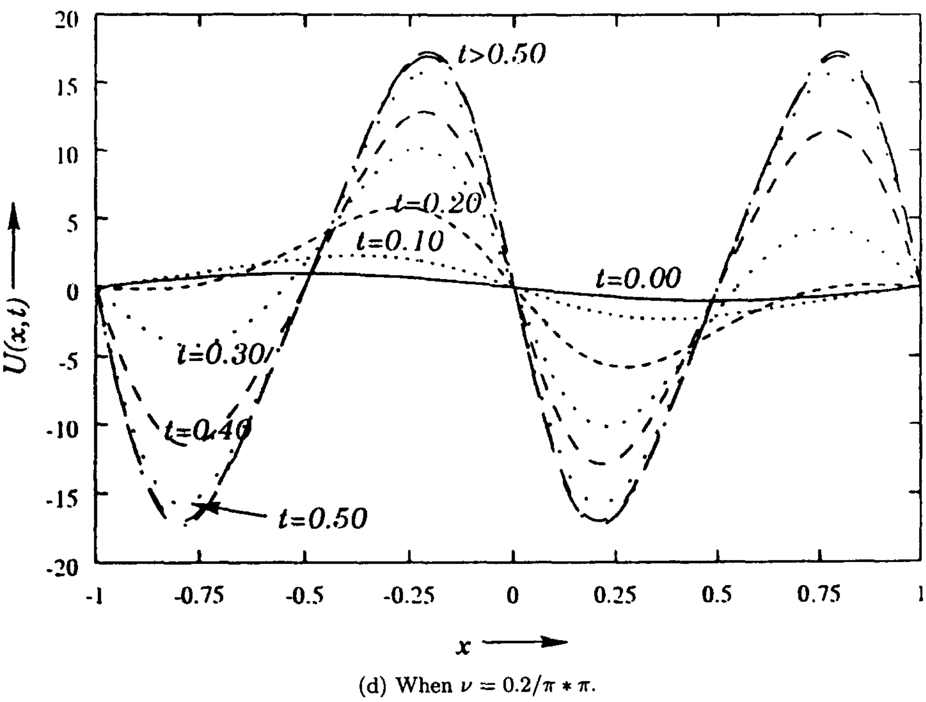
Figure 5. The graph of $U(x,t)$ of K-S equation for different ν.

We have solved the K-S equation mainly to validate the theoretical results and how well the numerical results mimic the qualitative behaviour of the K-S equation. For this purpose, we have used the software packages that are freely available from the web sites.

For the solution of the almost block diagonal linear system (4.11) and (4.12), we have employed the code ABDPACK [23,24]. This code has been especially developed for the solution of the system which arises from the orthogonal spline collocation methods using the monomial basis functions.



(c) When $\nu = 0.4/\pi * \pi$.

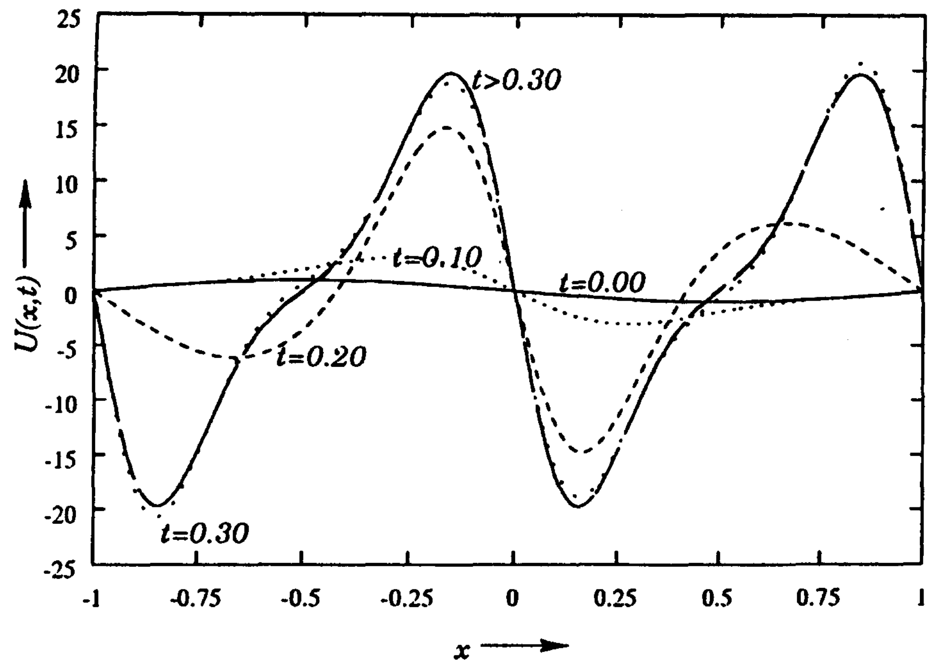


(d) When $\nu = 0.2/\pi * \pi$.

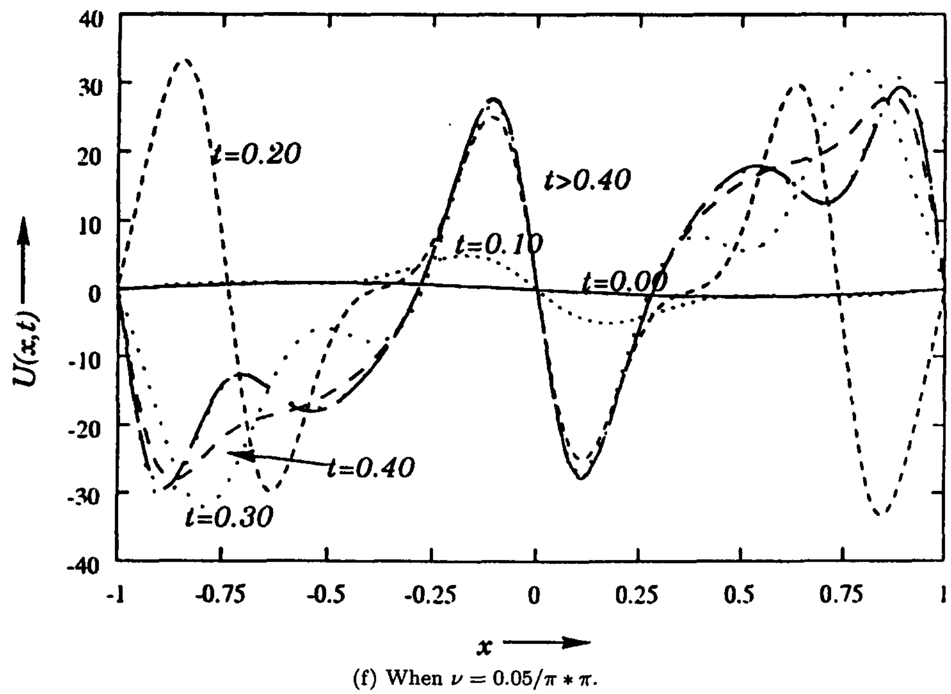
Figure 5. (cont.)

The DAE formulation of the K-S equation is presented in (4.9). Note that the *index* of the system is one [17,25]. We have used RADAU5, a software package that is based on a three stage implicit Runge-Kutta scheme of RADAU-IIA. This numerical scheme is self-starting and *stiffly accurate*. The solution obtained using RADAU5 has order of convergence five in every component. RADAU5 can also solve DAEs of index up to 3.

Although Robinson *et al.* [15] have solved a similar system which arises in the semidiscrete version of Schrödinger equation using the commercially available NAG library routine D02NNF that is based on a backward difference formula (BDF), we have chosen RADAU5 for the following



(e) When $\nu = 0.1/\pi * \pi$.



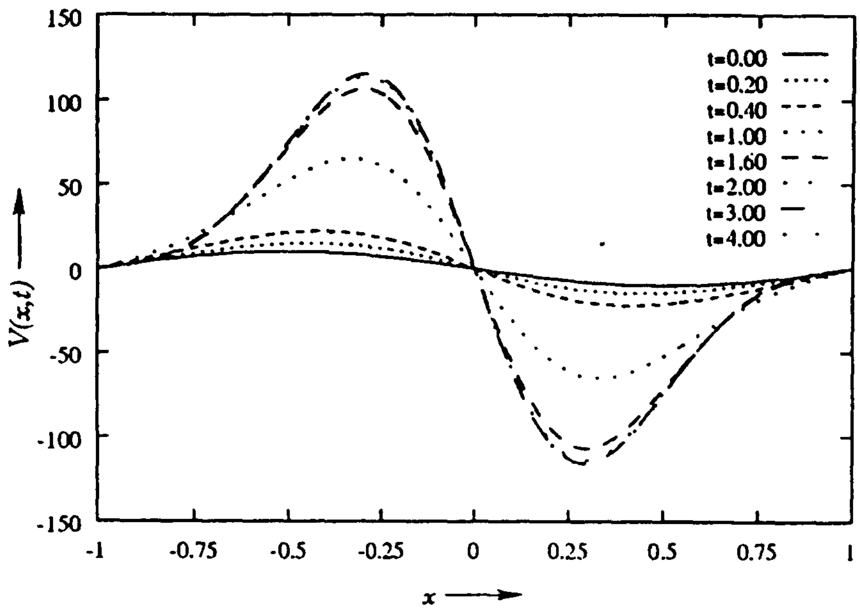
(f) When $\nu = 0.05/\pi * \pi$.

Figure 5. (cont.)

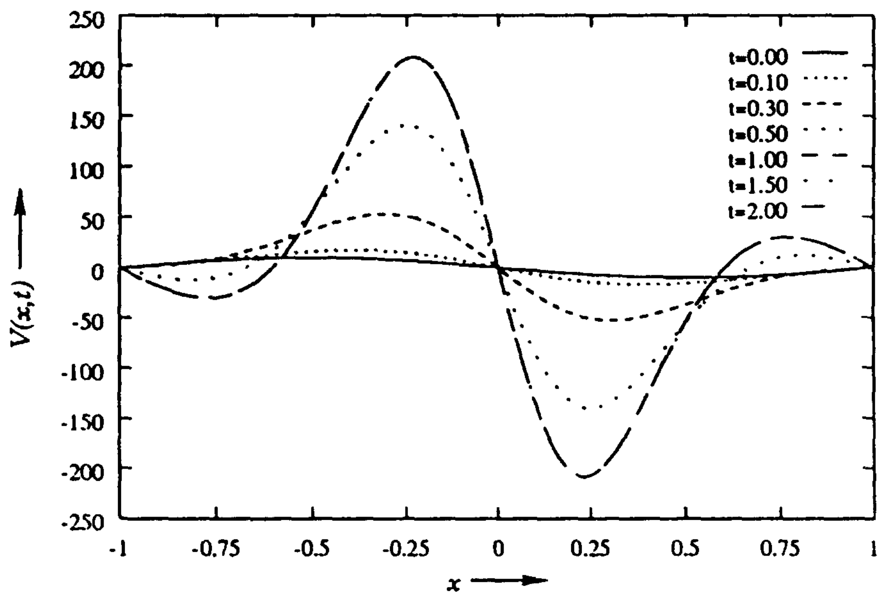
reason. As the BDF methods are unstable for steps greater than 6, one can only use methods with order up to 6. Although RADAU5 has order 5 only, this method can easily be extended for higher order methods. Indeed the newer version of this package, known as RADAUP, that has methods with order up to 13, is now available from Hairer and Wanner [18].

Now we describe the numerical experiments that has been conducted on the K-S equation. For computational convenience, we have considered the space in x direction as $[-1, 1]$ instead of [0,1]. The numerical experiments are carried out for (1.5)–(1.7) using the initial function

$$u(x, 0) = -\sin(\pi x), \qquad x \in (-1, 1).$$



(a) When $\nu = 0.8/\pi * \pi$.



(b) When $\nu = 0.5/\pi * \pi$.

Figure 6. The graph of $V(x,t)$ of K-S equation for different ν.

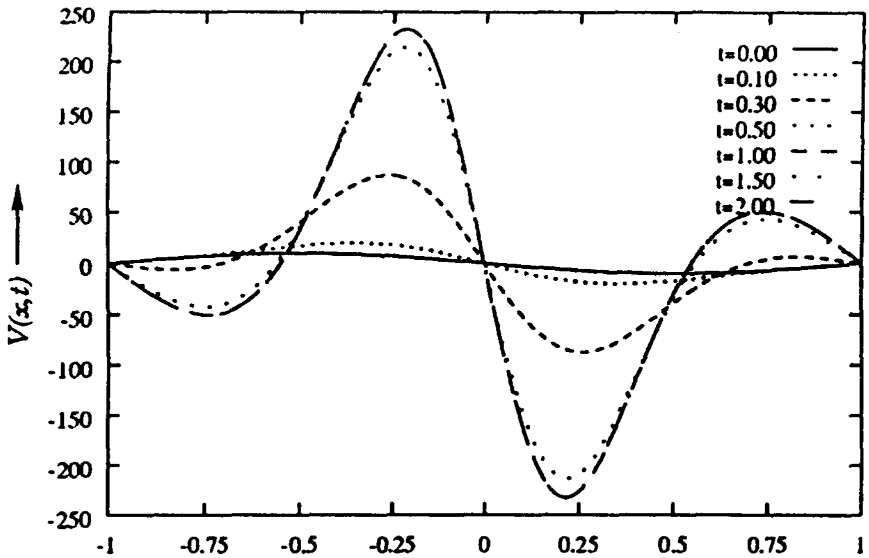
This is an odd, 2-periodic smooth function, satisfying the boundary conditions (1.7). All the theoretical results remain valid also in the new space. For example, the monotone property (2.4) is now valid for $\nu \geq 1/\pi^2$ instead of $\nu \geq 1/4\pi^2$ for all $t \geq 0$. Divide the domain into $N_i = 4, 8, 16, 32$ with each of equal intervals h_i, where

$$h_i = \frac{2}{N_i} \qquad i = 1, \dots, 4.$$

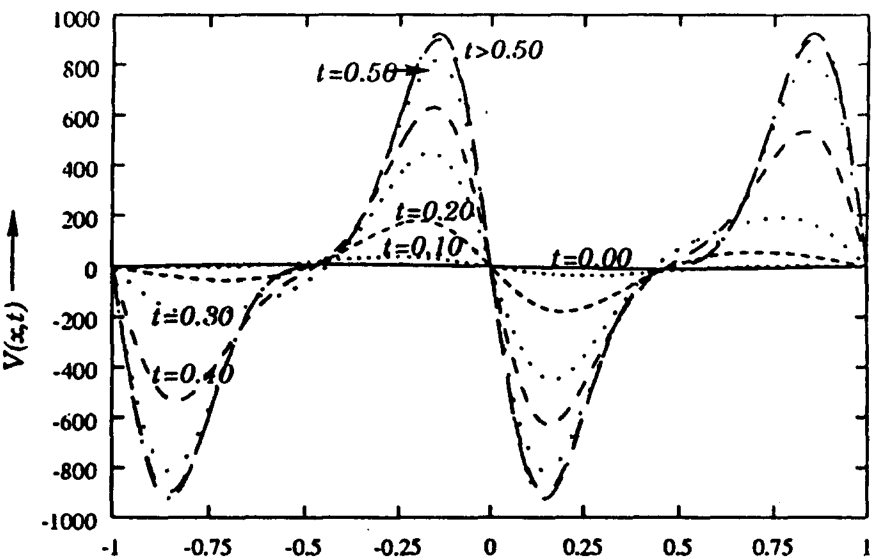
Since the exact solution of the K-S equation is not known, it has been replaced by numerical solution u_h with $h = 1/64$. The order of convergence for the numerical method has been computed by the formula

$$\text{order} = \frac{\log\left[\|u_h - u_{h_i}\|_{L^j} / \|u_h - u_{h_{i+1}}\|_{L^j}\right]}{\log(2)}, \qquad i = 1, 2, 3, \quad j = 2, \infty,$$

where u_{h_i} is the numerical solution with step size h_i and $h_{i+1} = h_i/2$.



(c) When $\nu = 0.4/\pi * \pi$.



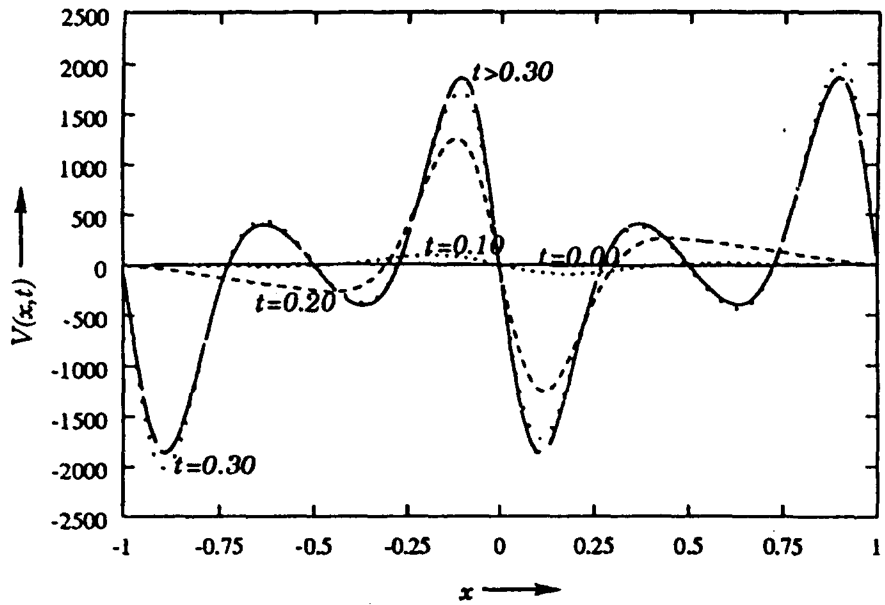
(d) When $\nu = 0.2/\pi * \pi$.

Figure 6. (cont.)

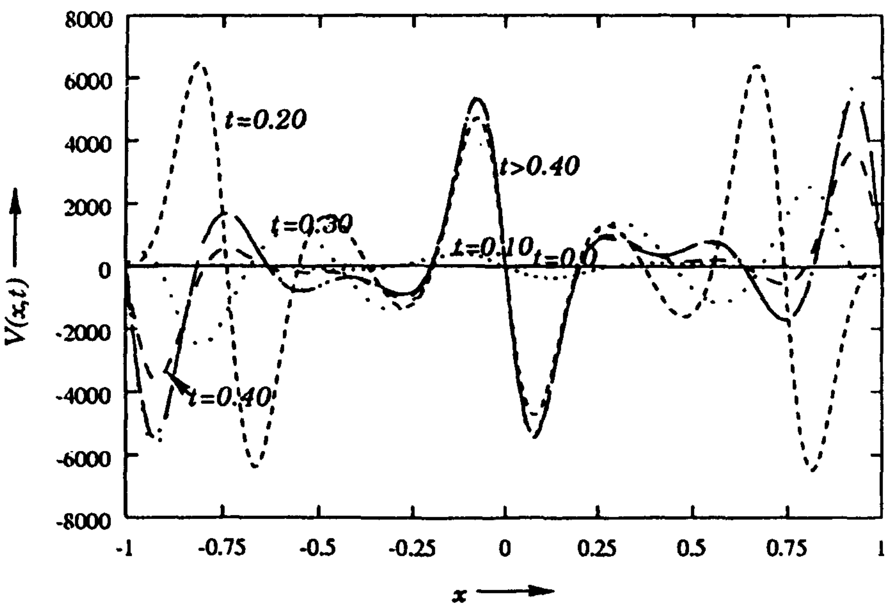
All the codes have been written in FORTRAN with double precision. The experiments have been conducted with the following parameter values for RADAU5: RTOL$= 10^{-6}$, ATOL$= 10^{-5}$, and the initial step size $k = 10^{-5}$. RADAU5 uses the variable step size method. The maximum step size used in all calculations is $k_{\max} = 0.087880501$. The numerical solution is carried out for several h_i with $\nu = 1.2/\pi^2$. In the following Tables 1 and 2, we observe that the order of convergence estimated numerically is approximately equal to 4. This confirms the theoretical order of convergence found in the Theorems 3.1 and 3.2.

4.1. Discussion and Conclusion

Below, we present some numerical experiments which inherently retain the important features of the original equation, namely; periodicity, symmetry, monotonicity, and dissipativity. When



(e) When $\nu = 0.1/\pi * \pi$.



(f) When $\nu = 0.05/\pi * \pi$.

Figure 6. (cont.)

Table 1. The order of convergence for $u(x,t)$ at $t = 2.00$.

N	$\|u_h - u_{h_i}\|_{L^2}$	Order	$\|u_h - u_{h_i}\|_{L^\infty}$	Order
4	0.2885555410113181E−02		0.1112469254850749E−02	
8	0.2198484300578422E−03	3.7143	0.9464817039302187E−04	3.5550
16	0.1421672425495744E−04	3.9508	0.6407943570197611E−05	3.8846
32	0.8704906524198726E−06	4.0296	0.3986857081382522E−06	4.0065

the initial function $u_0(x)$ is periodic with period, say L, then the same is true for the solution at any time $t > 0$, provided $\nu \geq 1/\pi^2$. This has been proved by Collet *et al.* [10] for $\nu = 1$. Our calculation shows that the periodicity is maintained for smaller ν values as well and this will

Table 2. The order of convergence for $\nu(x,t)$ at $t = 2.00$.

N	$\|v_h - v_{h_i}\|_{L^2}$	Order	$\|v_h - v_{h_i}\|_{L^\infty}$	Order
4	0.2753174205294434E−01		0.1063689851990920E−01	
8	0.2094856432480980E−02	3.7162	0.9059195497740924E−03	3.5536
16	0.1354412693774362E−03	3.9511	0.6128517819670165E−04	3.8858
32	0.8302116127022436E−05	4.0280	0.3815664612716319E−05	4.0055

be explained below. In Figures 1 and 2, the numerical solutions $U(x,t)$ and $V(x,t)$ are plotted versus x, respectively, for various time intervals, when $\nu = 1.2/\pi^2$.

The monotone property (2.4) of the K-S equation is preserved during the integration which is clear from Figure 3, i.e., the solution dies out fast as ν increases from $1/\pi^2$. Further, it is observed that for $\nu \geq 1/\pi^2$, the dissipative terms are entirely dominating during integration and that nonlinear effect is completely ignored. This property is called dissipativity which implies existence of an absorbing ball in the phase space attracting all the trajectories for large enough t. The conservation property of the K-S equation is maintained for $\nu = 1/\pi^2$.

For $\nu < 1/\pi^2$, the nonlinear term takes the lead over the dissipative term and the solution goes under tremendous distortion in a short duration of time. It is observed that the solution and its potential monotonically increase up to a certain time and then reach steady states. As ν is made smaller than $1/\pi^2$, although symmetric property around $x = 0$ is preserved, the solution retains periodic nature with varying periods, (see Figures 5a–5f). Moreover, as t increases beyond say $t = t^*$ (for $\nu = 0.8/\pi^2$, $t^* = 1.8$; $\nu = 0.5/\pi^2$, $t^* = 1.0$; $\nu = 0.4/\pi^2$, $t^* = 0.65$; $\nu = 0.2/\pi^2$, $t^* = 0.6$; $\nu = 0.1/\pi^2$, $t^* = 0.35$), all the solution trajectories coincide, which confirm existence of the absorbing balls (see Figures 5a–5f and 4). For $\nu = 0.05/\pi^2$, there is a steep increase in the solution followed by quite a bit of oscillations from $t = 0.2$ to $t^* = 0.5$ and then the trajectory is attracted towards the ball. Further decrease in ν (say, $\nu = 0.03/\pi^2$) results in a lot of oscillations (see Figure 4). These oscillations are more pronounced for the potential $V(x,t)$ (see Figures 6a–6f). Beyond this ν, the code breaks down due to the extremely small step size k. This is expected because for sufficiently small ν, the solution blows up as predicted by (2.3).

As we have applied Gronwall's lemma for *a priori* estimates in Theorem 3.1, the bounds depend on $\exp(T)$. Therefore, the theoretical results are not valid for large t. The above numerical experiments, however, suggest the existence of a maximal attractor. Some preliminary results for computing the inertial manifold are already available in literature. For example in [11,12], a spectral method in conjunction with nonlinear Galerkin technique is used to investigate the long time dynamics of the K-S equation. In the future, we shall address the question 'how well the inertial manifold is approximated by the discretization'.

Finally, we note that the order of convergence for the semidiscrete scheme can be increased by choosing C^1-piecewise polynomial of degree $r \geq 3$. Similarly, increasing the number of stages in RADAU-IIA (say, $s = 3, 5, 7$), it is possible to obtain higher order of convergence(say, $p = 5, 9, 13$) in the time direction.

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